SIAM J. NUMER. ANAL.
Vol. 43, No. 2, pp. 455–473

FOURTH-ORDER NONOSCILLATORY UPWIND AND CENTRAL SCHEMES FOR HYPERBOLIC CONSERVATION LAWS*

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Abstract. The aim of this work is to solve hyperbolic conservation laws by means of a finite volume method for both spatial and time discretization. We extend the ideas developed in [X.-D. Liu and S. Osher, *SIAM J. Numer. Anal.,* 33 (1996), pp. 760–779; X.-D. Liu and E. Tadmor, *Numer. Math.,* 79 (1998), pp. 397–425] to fourth-order upwind and central schemes. In order to do this, once we know the cell-averages of the solution, $\overline{u}_j^n$, in cells I_j at time $T = t^n$, we define a new three-degree reconstruction polynomial that in each cell, I_j, presents the same shape as the cell-averages $\{\overline{u}_{j-1}^n, \overline{u}_j^n, \overline{u}_{j+1}^n\}$. By combining this reconstruction with the nonoscillatory property and the maximum principle requirement described in [X.-D. Liu and S. Osher, *SIAM J. Numer. Anal.,* 33 (1996), pp. 760–779] we obtain a fourth-order scheme that satisfies the total variation bounded (TVB) property. Extension to systems is carried out by componentwise application of the scalar framework. Numerical experiments confirm the order of the schemes presented in this paper and their nonoscillatory behavior in different test problems.

Key words. central schemes, upwind schemes, high order, nonoscillatory, hyperbolic conservation laws

AMS subject classification. 65M06

DOI. 10.1137/S0036142903437106

1. Introduction.

In this paper we present three fourth-order numerical schemes in order to solve one-dimensional hyperbolic conservation laws

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad u(x,0) = u_0(x), \tag{1.1}$$

where $u_0(x)$ is a known bounded function.

Many of the high-order methods used to solve this problem employ an interpolating polynomial that reconstructs the pointvalues of the solution in terms of the cell-averages. There are two main types of schemes: upwind schemes and central schemes. Godunov-type schemes [7] are the forerunner for upwind schemes, which compute the cell-averages of the solution in the same spatial cells at all time steps. Similarly, Van Leer [25] presented a scheme with second-order accuracy in space and time. Later, Colella and Woodward [5] used two-degree polynomials, although their scheme satisfies the total variation diminishing (TVD) property, and, hence, it is limited to second order of accuracy in the L^1 norm. Harten et al. [8] introduced the essentially nonoscillatory (ENO) schemes, with an order of accuracy higher than two and able to capture sharp shocks without introducing oscillations. Similarly, different high-order numerical schemes have been developed, such as the weighted ENO (WENO) schemes (see Liu, Osher, and Chan [18], Jiang and Shu [10], or Balsara and Shu [2]). Extensions to multidimensional systems can also be found in Casper

*Received by the editors October 3, 2003; accepted for publication (in revised form) September 7, 2004; published electronically June 30, 2005.
http://www.siam.org/journals/sinum/43-2/43710.html

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and Atkins [4] or Balaguer et al. [1]. The schemes found in the latter references use the high-order Runge–Kutta schemes developed in Shu and Osher [23] for time integration, which maintain the spatial operator stability properties.

Although the first-order Lax–Friedrich scheme (see [6]) is probably the forerunner for central schemes, the central scheme of Nessyahu and Tadmor [21] has generated a significant number of works on high-resolution schemes that maintain the simplicity of the Riemann solver-free approach. The scheme developed in Nessyahu and Tadmor [21] has been extended to accuracy orders higher than 2 (see Liu and Tadmor [20], Jiang et al. [11], or Qiu and Shu [22]) and to several spatial dimensions (see Levy and Tadmor [15] and Jiang and Tadmor [12]). High-order central WENO schemes are described in Levy, Puppo, and Russo [16], [17].

We have focused our attention on the upwind scheme developed in Liu and Osher [19] and the central scheme described in Liu and Tadmor [20], which are third-order schemes, in the sense of local truncation error in regions without discontinuities. The algorithm developed in Liu and Osher [19] leads to a conservative scheme that satisfies the local maximum principle and guarantees that the number of extrema in the solution does not exceed the number of extrema of the initial condition $u_0(x)$. These properties allow achieving the total variation bounded (TVB) property. The approach used in that reference uses a simple centered stencil with quadratic reconstruction.

Liu and Tadmor [20] apply the procedure described in Liu and Osher [19] to central schemes and show the results obtained when solving differential equation systems. The resulting scheme is third-order accurate in space and time. In both references ([19] and [20]), time integration is performed using a finite volume method, approximating the resulting integrals with respect to time by a Gauss [19] or a Simpson [20] quadrature rule. The values of the solution at half time steps are approximated using a Taylor expansion. Jiang et al. [11] present a procedure to convert schemes which are based on staggered spatial grids into nonstaggered schemes, which are simpler to implement in frameworks which involve complex geometries and boundary conditions. However, it has been in some cases superseded by the semidiscrete central schemes (see Kurganov and Tadmor [13]) and their high-order extensions.

This paper presents an extension of the schemes developed in Liu and Osher [19] and Liu and Tadmor [20]. In contrast to them, our scheme is a fourth-order scheme in the sense of local truncation error. To this end, we will use a finite volume method with a conservative degree-three polynomial reconstruction that calculates the pointvalues of the solution from the cell-averages, by avoiding the increase in the number of solution extrema at the interior of each cell. This condition, together with the nonoscillatory property and the maximum principle requirement described in Liu and Osher [19], avoids spurious numerical oscillations in the computed solution.

The integrals respecting the two variables, space and time, are evaluated by means of a two-point Gauss quadrature. The values of the solution at half-time steps are calculated using a Taylor expansion with a fourth-order error, using the local Cauchy–Kowalewski procedure (see [8]) to approximate the time derivatives of the solution as a function of the derivatives with respect to x. We also present an extension to systems of equations, where the computed solution at quadrature nodes is obtained by the so called natural continuous extension of Runge–Kutta schemes (see Zennaro [26], Bianco, Puppo, and Russo [3], or Levy, Puppo, and Russo [16]).

In this paper, first, we present the equations that define the upwind and central schemes for solving the problem (1.1). Next, the fourth-order nonoscillatory reconstruction procedure is described. It guarantees that the resulting numerical scheme satisfies the properties that generate its nonoscillatory behavior. Finally, some problems with known analytical solution are solved to verify the order of the schemes presented here and to compare their behavior with the schemes developed in Liu and Osher [19] and Liu and Tadmor [20].

2. Upwind and central schemes. Let us suppose that the time interval is discretized uniformly into the values $t^n = n \cdot \Delta t,\ n = 0, 1, 2, \ldots, NT$. We assume that the grid points $\{x_j\}$ are distributed uniformly at the spatial domain at which (1.1) will be defined, verifying that $x_j = x_{j-1} + \Delta x\, \forall j = 1, \ldots, NX$, where Δx is a known constant. Given a point (x_j, t^n), we consider the control volume defined by $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times \left[t^n, t^{n+1}\right]$, where $x_{j\pm\frac{1}{2}} = x_j \pm \Delta x/2$. By integrating (1.1) over this control volume, we obtain

$$(2.1)\ \overline{u}_j^{n+1} = \overline{u}_j^n - \frac{1}{\Delta x}\left[\int_{t^n}^{t^{n+1}} f\left(u\left(x_{j+\frac{1}{2}}, \tau\right)\right) d\tau - \int_{t^n}^{t^{n+1}} f\left(u\left(x_{j-\frac{1}{2}}, \tau\right)\right) d\tau\right],$$

where the cell average $\overline{u}_j^n$ is defined as

$$(2.2) \qquad \overline{u}_j^n = \frac{1}{\Delta x}\int_{I_j} u\left(\varphi, t^n\right)\, d\varphi, \quad I_j = \left\{\varphi, |\varphi - x| \le \frac{\Delta x}{2}\right\}.$$

In (2.1) there is a relationship between the average values of the solution at the limit of the time interval, $\overline{u}_j^n$, $\overline{u}_j^{n+1}$, and its pointvalues at the boundary of the space interval, $u(x_{j\pm\frac{1}{2}}, \tau)$. The steps to follow in the implementation of numerical schemes can be described as follows.

(1) For each time value $t^n,\ n \in \{0, 1, \ldots, NT-1\}$, we have an approximation of the cell-averages of the solution $\overline{w}_j^n \cong \overline{u}_j^n\, \forall j \in \{0, 1, \ldots, NX\}$, at the nodes x_j. The approximation will be of order $O((\Delta x)^4)$.

(2) The pointvalues of $w(x, t^n)\, \forall x \in \{x_0 - \Delta x/2, \ldots, x_{NX} + \Delta x/2\}$ are reconstructed using a piecewise polynomial interpolation,

$$(2.3) \qquad w(x, t^n) \equiv \sum_{j=0}^{NX} R_j(x; \overline{w}^n)\chi_j(x), \quad \chi_j(x) = \begin{cases} 1 & \text{if} \quad x \in I_j, \\ 0 & \text{if} \quad x \notin I_j, \end{cases}$$

where $R_j(x; \overline{w}^n)$ is a polynomial that reconstructs the pointvalues of the solution using the discrete values $\overline{w}_i^n$, $i \in \{0, 1, \ldots, NX\}$, verifying

$$(2.4) \quad \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} R_j(x; \overline{w}^n) = \overline{w}_j^n, \quad R_j(x; \overline{w}^n) = w(x, t^n) + O((\Delta x)^4) \ \ \forall x \in I_j.$$

(3) In the case of the central schemes, the average values $\overline{w}_{j+\frac{1}{2}}^n$ are calculated using the approximation given in (2.3):

$$(2.5) \qquad \overline{w}_{j+\frac{1}{2}}^n \equiv \frac{1}{\Delta x}\left[\int_{x_j}^{x_j + \Delta x/2} R_j(x; \overline{w}^n)\, dx + \int_{x_j + \Delta x/2}^{x_{j+1}} R_{j+1}(x; \overline{w}^n)\, dx\right].$$

The integrals in (2.1) are evaluated in an exact way taking into account that $R_j(x; \overline{w}^n)$ and $R_{j+1}(x; \overline{w}^n)$ are degree-three polynomials.

(4) The integrals with respect to the time variable are approximated using a two-point Gauss quadrature. Thus,

$$\int_{t^n}^{t^{n+1}} f(w(x_{j\pm\frac{1}{2}},\tau))d\tau \approx \frac{\Delta t}{2}\left(f(w(x_{j\pm\frac{1}{2}},t^n+\beta_0))+f(w(x_{j\pm\frac{1}{2}},t^n+\beta_1))\right) \tag{2.6}$$

$$\beta_0 = \Delta t\left(\frac{1-1/\sqrt{3}}{2}\right),\quad \beta_1 = \Delta t\left(\frac{1+1/\sqrt{3}}{2}\right). \tag{2.7}$$

In order to approximate the pointvalues of w at the time steps that appear in (2.6), we may use a Taylor expansion with an error $O((\Delta x^4))$. This technique is used, for example, in Liu and Osher [19] and Liu and Tadmor [20]. Another efficient method would be the natural continuous extension of Runge–Kutta methods advocated by Bianco, Puppo, and Russo [3] and Levy, Puppo, and Russo [16]. We will use this method in the resolution of systems of equations which achieve the same accuracy with much lower computational effort.

(5) In order to calculate the cell-averages of w at t^{n+1}, we distinguish two cases.

(5a) *Upwind schemes.* The cells are intervals centered at each x_j (equation (2.1), after replacing the function u—in that equation—for the function w). In order to calculate the value of $f(w(x_{j\pm\frac{1}{2}},t^n+\beta_k))$ in expression (2.6), we will use the Roe flux with entropy fix, although other fluxes can also be used, as those described in Liu and Osher [19].

(5b) *Central schemes.* The cells are intervals centered at each $x_{j+\frac{1}{2}}$.

$$\overline{w}^{n+1}_{j+\frac{1}{2}} = \overline{w}^n_{j+\frac{1}{2}} - \frac{1}{\Delta x}\left[\int_{t^n}^{t^{n+1}} f\left(w\left(x_{j+1},\tau\right)\right)d\tau - \int_{t^n}^{t^{n+1}} f\left(w\left(x_j,\tau\right)\right)d\tau\right]. \tag{2.8}$$

The terms on the right-hand side in (2.1) and (2.8) are calculated using the approximations (2.5)–(2.7).

(6) We go back to step (1) and restart the procedure until calculating $\overline{w}^{NT}_i \cong \overline{u}(x_i,t^{NT})$, $i\in\{0,1,\dots,NX\}$. Then, we use formula (2.3) to obtain the pointvalues with $O((\Delta x)^4)$.

3. Fourth-order nonoscillatory reconstruction. This section presents the reconstruction procedure used to obtain each $R_j(x;\overline{w}^n)$ from the cell averages $\overline{w}^n_k$, $k\in\{j-2,j-1,j,j+1,j+2\}$.

3.1. Fourth order and conservation. Initially, we will consider the degree-three polynomial that verifies these conditions:

$$p_j(x_j;\overline{w}^n)=\overline{w}^n_j,\quad p_j(x_{j-1};\overline{w}^n)=\overline{w}^n_{j-1},\ \ p_j(x_{j+1};\overline{w}^n)=\overline{w}^n_{j+1}, \tag{3.1}$$

$$\Delta x\frac{dp_j}{dx}(x_j;\overline{w}^n)=\Delta x\frac{\partial\overline{w}}{\partial x}(x_j,t^n)\equiv d^n_j,\quad \text{where } \overline{w}(x,t^n)=\frac{1}{\Delta x}\int_{x-\Delta x/2}^{x+\Delta x/2} w(\varphi,t^n)\,d\varphi.$$

This polynomial can be expressed as

$$\begin{aligned} p_j\left(x;\overline{w}^n\right) &= \overline{w}^n_j + d^n_j\cdot\left(\frac{x-x_j}{\Delta x}\right)+\left(\frac{\overline{w}^n_{j-1}-2\overline{w}^n_j+\overline{w}^n_{j+1}}{2}\right)\cdot\left(\frac{x-x_j}{\Delta x}\right)^2 \\ &\quad + \left(\frac{-\overline{w}^n_{j-1}+\overline{w}^n_{j+1}-2d^n_j}{2}\right)\cdot\left(\frac{x-x_j}{\Delta x}\right)^3. \end{aligned} \tag{3.2}$$

Since

$$w(x,t^n) = \overline{w}(x,t^n) - \frac{1}{24}(\Delta x)^2 \frac{\partial^2 \overline{w}(x,t^n)}{\partial x^2} + O(\Delta x)^4, \tag{3.3}$$

the conservative polynomial, $q_j(x;\overline{w}^n)$ that verifies the conditions in (2.4) can be defined as

$$q_j(x;\overline{w}^n) = p_j(x;\overline{w}^n) - \frac{1}{24}(\Delta x)^2 \frac{d^2 p_j(x;\overline{w}^n)}{dx^2}. \tag{3.4}$$

Therefore,

$$q_j(x;\overline{w}^n) = \overline{w}_j^n - \frac{1}{24}\left(\overline{w}_{j-1}^n - 2\overline{w}_j^n + \overline{w}_{j+1}^n\right) + \left(\frac{\overline{w}_{j-1}^n - \overline{w}_{j+1}^n + 10d_j^n}{8}\right)\left(\frac{x-x_j}{\Delta x}\right)$$

$$+\left(\frac{\overline{w}_{j-1}^n - 2\overline{w}_j^n + \overline{w}_{j+1}^n}{2}\right)\left(\frac{x-x_j}{\Delta x}\right)^2 + \left(\frac{-\overline{w}_{j-1}^n + \overline{w}_{j+1}^n - 2d_j^n}{2}\right)\left(\frac{x-x_j}{\Delta x}\right)^3. \tag{3.5}$$

In case that

$$d_j^n = ds_j^n \equiv \frac{2}{3}\overline{w}_{j+1}^n - \frac{2}{3}\overline{w}_{j-1}^n - \frac{1}{12}\overline{w}_{j+2}^n + \frac{1}{12}\overline{w}_{j-2}^n, \tag{3.6}$$

$q_j(x;\overline{w}^n)$ coincides with the centered polynomial, defined as the average value between two conservative piecewise polynomials: the conservative polynomial which uses $\{\overline{w}_{j-1}^n, \overline{w}_j^n, \overline{w}_{j+1}^n, \overline{w}_{j+2}^n\}$ and the polynomial based on $\{\overline{w}_{j-2}^n, \overline{w}_{j-1}^n, \overline{w}_j^n, \overline{w}_{j+1}^n\}$. Then, by replacing the value of d_j^n given in (3.6) in expression (3.5), we obtain the following conservative polynomial that verifies conditions (2.4):

$$q_j^*(x;\overline{w}^n) = C_{o,j}^n + C_{1,j}^n\left(\frac{x-x_j}{\Delta x}\right) + C_{2,j}^n\left(\frac{x-x_j}{\Delta x}\right)^2 + C_{3,j}^n\left(\frac{x-x_j}{\Delta x}\right)^3, \tag{3.7}$$

$$C_{o,j}^n = \overline{w}_j^n - \frac{1}{24}\left(\overline{w}_{j+1}^n - 2\overline{w}_j^n + \overline{w}_{j-1}^n\right), C_{1,j}^n = \frac{-5\overline{w}_{j+2}^n + 34\overline{w}_{j+1}^n - 34\overline{w}_{j-1}^n + 5\overline{w}_{j-2}^n}{48},$$

$$C_{2,j}^n = \frac{1}{2}\left(\overline{w}_{j+1}^n - 2\overline{w}_j^n + \overline{w}_{j-1}^n\right),\ C_{3,j}^n = \frac{1}{12}\left(\overline{w}_{j+2}^n - 2\overline{w}_{j+1}^n + 2\overline{w}_{j-1}^n - \overline{w}_{j-2}^n\right).$$

3.2. Shape-preserving when the cell-averages form a monotone sequence. We will define d_j^n in (3.5) so that if the cell-averages $\{\overline{w}_{j-1}^n, \overline{w}_j^n, \overline{w}_{j+1}^n\}$ form a monotone sequence, then $q_j(x;\overline{w}^n)$ is monotone on I_j. We will denote as shape-preserving properties the following:

(I) $q_j(x;\overline{w}^n)$ is monotonically increasing in I_j if $\overline{w}_{j-1}^n \le \overline{w}_j^n \le \overline{w}_{j+1}^n$.

(II) $q_j(x;\overline{w}^n)$ is monotonically decreasing in I_j if $\overline{w}_{j-1}^n \ge \overline{w}_j^n \ge \overline{w}_{j+1}^n$.

To simplify the notation, first we define

$$\overline{W}C_j^n = \overline{w}_{j+1}^n - \overline{w}_{j-1}^n,\ \overline{W}R_j^n = \overline{w}_{j+1}^n - \overline{w}_j^n,\ \overline{W}C2_j^n = \overline{w}_{j+2}^n - \overline{w}_{j-2}^n. \tag{3.8}$$

Observation 1. If $d_j^n = \frac{\overline{W}C_j^n}{2}$, then according to (3.5) $q_j(x;\overline{w}^n)$ coincides with a quadratic polynomial. In this case, $\frac{dq_j(x_j \pm \Delta x/2;\overline{w}^n)}{dx} = \pm\frac{\overline{w}_{j\pm1}^n - \overline{w}_j^n}{\Delta x}$, and therefore the shape-preserving properties are verified.

Observation 2. In the case at which $d_j^n = ds_j^n$ (defined in (3.6)), the following hold.

1. If $\left(2 \cdot \overline{W}C_j^n = \overline{W}C2_j^n\right)$, then $d_j^n = \frac{\overline{W}C_j^n}{2}$ (see Observation 1).

2. If $\left(2 \cdot \overline{W}C_j^n > \overline{W}C2_j^n\right)$ and $\overline{w}_{j-1}^n \leq \overline{w}_j^n \leq \overline{w}_{j+1}^n$, then, according to (3.7),

$$\frac{d^3 q_j^*(x;\overline{w}^n)}{dx^3} = \frac{1}{(\Delta x)^3}\left(6 \cdot C_{3,j}^n\right) = \frac{1}{2 \cdot (\Delta x)^3}\left(\overline{W}C2_j^n - 2 \cdot \overline{W}C_j^n\right) < 0,$$

and thus $\frac{d(q_j^*(x;\overline{w}^n))}{dx}$ achieves the minimum value at the endpoints of the interval under consideration. Since

$$\begin{aligned} \frac{dq_j^*(x_j + \Delta x/2;\overline{w}^n)}{dx} &= \frac{1}{24\Delta x}\left(-\overline{W}C2_j^n + 2 \cdot \overline{W}C_j^n + 24 \cdot \overline{W}R_j^n\right) > 0, \\ \frac{dq_j^*(x_j - \Delta x/2;\overline{w}^n)}{dx} &= \frac{1}{24\Delta x}\left(-\overline{W}C2_j^n + 26 \cdot \overline{W}C_j^n - 24 \cdot \overline{W}R_j^n\right) > 0, \end{aligned} \tag{3.9}$$

$\mathrm{Min}\{\frac{d(q_j^*(x;\overline{w}^n))}{dx}\ \forall x \in I_j\} > 0$ and $q_j^*(x;\overline{w}^n)$ is monotonically increasing in I_j.

3. If $\left(2 \cdot \overline{W}C_j^n < \overline{W}C2_j^n\right)$ and $\overline{w}_{j-1}^n \leq \overline{w}_j^n \leq \overline{w}_{j+1}^n$, then the derivative of $q_j^*(x;\overline{w}^n)$, defined in (3.7), has a minimum at point

$$x_{MI} = x_j + \frac{\Delta x}{3}\left(\frac{2 \cdot \overline{W}R_j^n - \overline{W}C_j^n}{(1/6)\left(2 \cdot \overline{W}C_j^n - \overline{W}C2_j^n\right)}\right) = x_j - 4 \cdot \Delta x\left(\frac{\overline{W}R_j^n - (1/2)\overline{W}C_j^n}{\left(\overline{W}C2_j^n - 2 \cdot \overline{W}C_j^n\right)}\right).$$

In this way, if $\left|\overline{W}R_j^n - \frac{1}{2}\overline{W}C_j^n\right| \geq \frac{1}{8}\left|\overline{W}C2_j^n - 2 \cdot \overline{W}C_j^n\right|$, then

$$\left(\overline{W}R_j^n > \frac{1}{2}\overline{W}C_j^n \Rightarrow x_{MI} \leq x_j - \frac{\Delta x}{2}\right) \text{ and } \left(\overline{W}R_j^n < \frac{1}{2}\overline{W}C_j^n \Rightarrow x_{MI} \geq x_j + \frac{\Delta x}{2}\right).$$

Thus, the minimum $\mathrm{Min}\{\frac{d(q_j^*(x;\overline{w}^n))}{dx}\ \forall x \in I_j\}$ is achieved at one of these boundary points, $x = x_j \pm \frac{\Delta x}{2}$. However, in this case we cannot ensure that the inequalities given in (3.9) are always verified.

4. If $\left(2 \cdot \overline{W}C_j^n < \overline{W}C2_j^n\right)$ and $\overline{w}_{j-1}^n \geq \overline{w}_j^n \geq \overline{w}_{j+1}^n$, then we can prove that $q_j^*(x;\overline{w}^n)$ is monotonically decreasing in I_j.

5. If $\left(2 \cdot \overline{W}C_j^n > \overline{W}C2_j^n\right)$ and $\overline{w}_{j-1}^n \geq \overline{w}_j^n \geq \overline{w}_{j+1}^n$, then we can prove that $|x_{MI} - x_j| \geq \Delta x/2$ when $\left|\overline{W}R_j^n - \frac{1}{2}\overline{W}C_j^n\right| \geq \frac{1}{8}\left|\overline{W}C2_j^n - 2 \cdot \overline{W}C_j^n\right|$, but we cannot ensure that $q_j^*(x;\overline{w}^n)$ is always monotonically decreasing in I_j.

Observation 3. Supposing that $2 \cdot d_j^n < \overline{W}C_j^n$, then the derivative of $q_j(x;\overline{w}^n)$, defined in (3.5), has a minimum at point

$$x_{MI} = x_j + \frac{\Delta x}{3}\left(\frac{2 \cdot \overline{W}R_j^n - \overline{W}C_j^n}{2 \cdot d_j^n - \overline{W}C_j^n}\right).$$

In addition,

$$\frac{dq_j(x_{MI};\overline{w}^n)}{dx} = q_{xj}^n(d_j^n) \equiv \frac{1}{8\Delta x}\left(10 \cdot d_j^n - \overline{W}C_j^n\right) + \frac{1}{6\Delta x}\left(\frac{\left(2 \cdot \overline{W}R_j^n - \overline{W}C_j^n\right)^2}{\left(2 \cdot d_j^n - \overline{W}C_j^n\right)}\right).$$

This is a function that depends on d_j^n and coincides with a hyperbola. In it, the value of $d_j^n = \frac{\overline{W}C_j^n}{2} - S_j^n \frac{\sqrt{15}}{15}\left|2 \cdot \overline{W}R_j^n - \overline{W}C_j^n\right|$ is a local maximum of $q_{xj}^n(d_j^n)$ when $S_j^n > 0$ and a local minimum of $q_{xj}^n(d_j^n)$ when $S_j^n < 0$ being $S_j^n = \mathrm{Sign}(\overline{W}C_j^n)$.

3.2.1. Definition of d_j^n. The polynomial $q_j^*(x;\overline{w}^n)$ defined in (3.7) does not fulfill the shape-preserving properties defined in this subsection for any sequence of values $\{\overline{w}_{j-2}^n, \overline{w}_{j-1}^n, \overline{w}_j^n, \overline{w}_{j+1}^n, \overline{w}_{j+2}^n\}$. Therefore, we have to define a procedure that adequately defines the slopes d_j^n.

We will consider the value of $d_j^n = ds_j^n$ (given in (3.6)) except when this shape-preserving property is not fulfilled. For this, we use the notation

$$ds1_j^n = \frac{\overline{W}C_j^n}{10},\ ds2_j^n = \frac{1}{2}\left(\overline{W}C_j^n - 4\cdot\overline{W}R_j^n\right),\ ds3_j^n = \frac{1}{2}\left(4\cdot\overline{W}R_j^n - 3\cdot\overline{W}C_j^n\right),$$

$$S_j^n = \mathrm{Sign}(\overline{W}C_j^n),\ C1 = \frac{\sqrt{15}}{15},\ C2 = \frac{15-\sqrt{15}}{28} \tag{3.10}$$

and define d_j^n in the following way:

(A1) If $S_j^n = 0$, then $d_j^n = 0$.

(A2) If $S_j^n \neq 0$ and $\left(2\cdot S_j^n\cdot\overline{W}C_j^n \geq S_j^n\cdot\overline{W}C2_j^n\right)$, then $d_j^n = ds_j^n$.

(A3) If $S_j^n \neq 0$ and $\left(2\cdot S_j^n\cdot\overline{W}C_j^n < S_j^n\cdot\overline{W}C2_j^n\right)$, then the following hold:

(A3.1) If $\overline{w}_j^n = \frac{\overline{w}_{j+1}^n + \overline{w}_{j-1}^n}{2}$, we define

$$d_j^n = \begin{cases} \mathrm{Max}\left\{ds1_j^n, ds_j^n\right\} & \text{if } S_j^n > 0,\\ \mathrm{Min}\left\{ds1_j^n, ds_j^n\right\} & \text{if } S_j^n < 0.\end{cases}$$

(A3.2) If $\overline{w}_j^n \neq \frac{\overline{w}_{j+1}^n + \overline{w}_{j-1}^n}{2}$, then the following hold:

(A3.2.1) If $\left|\overline{W}R_j^n - \frac{1}{2}\overline{W}C_j^n\right| \geq \frac{1}{8}\left|\overline{W}C2_j^n - 2\cdot\overline{W}C_j^n\right|$, then

$$d_j^n = \begin{cases} \mathrm{Max}\left\{ds2_j^n,\ ds3_j^n,\ ds_j^n\right\} & \text{if } S_j^n > 0,\\ \mathrm{Min}\left\{ds2_j^n,\ ds3_j^n,\ ds_j^n\right\} & \text{if } S_j^n < 0.\end{cases}$$

(A3.2.2) If $\left|\overline{W}R_j^n - \frac{1}{2}\overline{W}C_j^n\right| < \frac{1}{8}\left|\overline{W}C2_j^n - 2\cdot\overline{W}C_j^n\right|$, then

$$d_j^n = \begin{cases} \frac{\overline{W}C_j^n}{2} - S_j^n\cdot C1\cdot\left|2\cdot\overline{W}R_j^n - \overline{W}C_j^n\right| & \text{if } \left|\frac{\overline{W}R_j^n}{\overline{W}C_j^n} - \frac{1}{2}\right| \leq C2,\\ \frac{\overline{W}C_j^n}{2} & \text{if } \left|\frac{\overline{W}R_j^n}{\overline{W}C_j^n} - \frac{1}{2}\right| > C2.\end{cases}$$

THEOREM 3.1. *With this definition of d_j^n the polynomial $q_j(x;\overline{w}^n)$ defined by means of* (3.5) *verifies the following shape-preserving properties:*

(I) $q_j(x;\overline{w}^n)$ *is monotonically increasing in* I_j *if* $\overline{w}_{j-1}^n \leq \overline{w}_j^n \leq \overline{w}_{j+1}^n$.

(II) $q_j(x;\overline{w}^n)$ *is monotonically decreasing in* I_j *if* $\overline{w}_{j-1}^n \geq \overline{w}_j^n \geq \overline{w}_{j+1}^n$.

Proof. Let us suppose that $\overline{w}_{j-1}^n \leq \overline{w}_j^n \leq \overline{w}_{j+1}^n$. We have to prove that $q_j(x;\overline{w}^n)$ is monotonically increasing in $I_j = \left[x_j - \frac{\Delta x}{2}, x_j + \frac{\Delta x}{2}\right]$. A similar argument allows us to prove that $q_j(x;\overline{w}^n)$ is monotonically decreasing in I_j when $\overline{w}_{j-1}^n \geq \overline{w}_j^n \geq \overline{w}_{j+1}^n$. We will consider all the possible cases.

Case (A1). $S_j^n = 0$, that is, $\overline{w}_{j-1}^n = \overline{w}_j^n = \overline{w}_{j+1}^n$. Then $d_j^n = 0$ and $q_j(x;\overline{w}^n) = \overline{w}_j^n$. This polynomial has the same degree of monotonicity as the cell averages $\{\overline{w}_{j-1}^n, \overline{w}_j^n, \overline{w}_{j+1}^n\}$.

Case (A2). $S_j^n = 1$ and $\left(2\cdot\overline{W}C_j^n \geq \overline{W}C2_j^n\right)$. In this case, $d_j^n = ds_j^n$ and, after observation 2, $q_j(x;\overline{w}^n)$ is monotonically increasing in I_j.

Case (A3.1). $S_j^n = 1$, $\left(2 \cdot \overline{W}C_j^n < \overline{W}C2_j^n\right)$, and $\overline{w}_j^n = \frac{\overline{w}_{j+1}^n + \overline{w}_{j-1}^n}{2}$. Then $(2 \cdot \overline{W}R_j^n = \overline{W}C_j^n)$ and according to Observation 3,

$$x_{MI} = x_j, \quad \frac{dq_j(x_{MI}; \overline{w}^n)}{dx} = \frac{1}{8\Delta x}\left(10 \cdot d_j^n - \overline{W}C_j^n\right).$$

Since in this case $d_j^n = \text{Max}\{\frac{\overline{W}C_j^n}{10}, ds_j^n\}$, then $\frac{dq_j(x_{MI};\overline{w}^n)}{dx} \geq 0$ and $q_j(x;\overline{w}^n)$ is monotonically increasing in I_j.

Case (A3.2.1). $S_j^n = 1$, $\left(2 \cdot \overline{W}C_j^n < \overline{W}C2_j^n\right)$, $\overline{w}_j^n \neq \frac{\overline{w}_{j+1}^n + \overline{w}_{j-1}^n}{2}$, and $\left|\overline{W}R_j^n - \frac{1}{2}\overline{W}C_j^n\right| \geq \frac{1}{8}\left|\overline{W}C2_j^n - 2 \cdot \overline{W}C_j^n\right|$. In this case, $d_j^n = \text{Max}\{\frac{1}{2}(\overline{W}C_j^n - 4 \cdot \overline{W}R_j^n), \frac{1}{2}(4 \cdot \overline{W}R_j^n - 3 \cdot \overline{W}C_j^n), ds_j^n\}$. Since $\left(2 \cdot \overline{W}C_j^n < \overline{W}C2_j^n\right)$, then $2 \cdot ds_j^n < \overline{W}C_j^n$. In addition,

$$\overline{W}R_j^n = 0 \Rightarrow d_j^n = \frac{\overline{W}C_j^n}{2}, \; \overline{W}R_j^n = \overline{W}C_j^n \Rightarrow d_j^n = \frac{\overline{W}C_j^n}{2}.$$

If $d_j^n = \frac{\overline{W}C_j^n}{2}$, it follows that $q_j(x;\overline{w}^n)$ coincides with a quadratic polynomial, which is monotonically increasing in I_j, as we have seen in Observation 1. Therefore, we can suppose that $0 < \overline{W}R_j^n < \overline{W}C_j^n$. Hence

$$\left(\overline{W}C_j^n - 4 \cdot \overline{W}R_j^n\right) < \overline{W}C_j^n, \; \overline{W}C_j^n - \left(4 \cdot \overline{W}R_j^n - 3 \cdot \overline{W}C_j^n\right) = 4 \cdot \left(\overline{W}C_j^n - \overline{W}R_j^n\right) > 0$$

so that it follows that $(2 \cdot d_j^n < \overline{W}C_j^n)$. On the other hand, applying Observations 2 and 3,

$$\left.\begin{array}{l} d_j^n \geq ds_j^n \\ \overline{W}R_j^n > \frac{1}{2}\overline{W}C_j^n \end{array}\right\} \Rightarrow x_{MI} = x_j + \frac{\Delta x}{3}\left(\frac{2 \cdot \overline{W}R_j^n - \overline{W}C_j^n}{2 \cdot d_j^n - \overline{W}C_j^n}\right) \leq x_j + \frac{\Delta x}{3}\left(\frac{2 \cdot \overline{W}R_j^n - \overline{W}C_j^n}{2 \cdot ds_j^n - \overline{W}C_j^n}\right)$$
$$\leq x_j - \frac{\Delta x}{2},$$
$$\left.\begin{array}{l} d_j^n \geq ds_j^n \\ \overline{W}R_j^n < \frac{1}{2}\overline{W}C_j^n \end{array}\right\} \Rightarrow x_{MI} = x_j - \frac{\Delta x}{3}\left(\frac{\overline{W}C_j^n - 2 \cdot \overline{W}R_j^n}{2 \cdot d_j^n - \overline{W}C_j^n}\right) \geq x_j - \frac{\Delta x}{3}\left(\frac{\overline{W}C_j^n - 2 \cdot \overline{W}R_j^n}{2 \cdot ds_j^n - \overline{W}C_j^n}\right)$$
$$\geq x_j + \frac{\Delta x}{2}.$$

In this way, because of $d_j^n \geq ds_j^n$ we deduce the following:

$$x_{MI} \notin \left]x_j - \frac{\Delta x}{2}, x_j + \frac{\Delta x}{2}\right[.$$

Thus, the minimum, $\text{Min}\{\frac{d(q_j(x;\overline{w}^n))}{dx} \, \forall x \in I_j\}$, is achieved at one of the boundary points $x = x_j \pm \frac{\Delta x}{2}$. By derivating in formula (3.5), we get that

$$\frac{dq_j(x_j + \Delta x/2; \overline{w}^n)}{dx} \geq 0 \Longleftrightarrow d_j^n \geq \frac{1}{2}\left(\overline{W}C_j^n - 4 \cdot \overline{W}R_j^n\right),$$
$$\frac{dq_j(x_j - \Delta x/2; \overline{w}^n)}{dx} \geq 0 \Longleftrightarrow d_j^n \geq \frac{1}{2}\left(4 \cdot \overline{W}R_j^n - 3 \cdot \overline{W}C_j^n\right).$$

The definition of d_j^n allows us to state that $\text{Min}\{\frac{d(q_j(x;\overline{w}^n))}{dx} \, \forall x \in I_j\} \geq 0$, and thus we conclude that $q_j(x;\overline{w}^n)$ is monotonically increasing in I_j.

Case (A3.2.2). $S_j^n = 1$, $\left(2 \cdot \overline{W}C_j^n < \overline{W}C2_j^n\right)$, $\overline{w}_j^n \neq \frac{\overline{w}_{j+1}^n + \overline{w}_{j-1}^n}{2}$, and $\left|\overline{W}R_j^n - \frac{1}{2}\overline{W}C_j^n\right| < \frac{1}{8}\left|\overline{W}C2_j^n - 2 \cdot \overline{W}C_j^n\right|$. In this case,

$$d_j^n = \begin{cases} \frac{\overline{W}C_j^n}{2} - \frac{\sqrt{15}}{15}\left|2 \cdot \overline{W}R_j^n - \overline{W}C_j^n\right| & \text{if } \left|\frac{\overline{W}R_j^n}{\overline{W}C_j^n} - \frac{1}{2}\right| \leq \frac{15 - \sqrt{15}}{28}, \\ \frac{\overline{W}C_j^n}{2} & \text{if } \left|\frac{\overline{W}R_j^n}{\overline{W}C_j^n} - \frac{1}{2}\right| > \frac{15 - \sqrt{15}}{28} \end{cases}$$

so that it follows that $2\cdot d_j^n \le \overline{W}C_j^n$. In the case in which $2\cdot d_j^n = \overline{W}C_j^n$ it follows that $q_j(x;\overline{w}^n)$ coincides with a two-degree polynomial, which is monotonically increasing in I_j, as we have seen in Observation 1. Therefore, we can suppose that $2\cdot d_j^n < \overline{W}C_j^n$ and $|\frac{\overline{W}R_j^n}{\overline{W}C_j^n} - \frac{1}{2}| \le \frac{15-\sqrt{15}}{28}$. In this situation, $\frac{dq_j(x;\overline{w}^n)}{dx}$ reaches a minimum at point $x_{MI} = x_j + \frac{\Delta x}{3}(\frac{2\cdot\overline{W}R_j^n - \overline{W}C_j^n}{2\cdot d_j^n - \overline{W}C_j^n})$ (see Observation 3). Given that

$$\overline{W}R_j^n > \frac{1}{2}\overline{W}C_j^n \Rightarrow \left\{\left(\frac{5}{6}\,\overline{W}C_j^n - \frac{2}{3}\overline{W}R_j^n\right) < \frac{\overline{W}C_j^n}{2} - \frac{\sqrt{15}}{15}\left(2\cdot\overline{W}R_j^n - \overline{W}C_j^n\right) = d_j^n\right\},$$

$$\overline{W}R_j^n < \frac{1}{2}\overline{W}C_j^n \Rightarrow \left\{\left(\frac{1}{6}\,\overline{W}C_j^n + \frac{2}{3}\overline{W}R_j^n\right) < \frac{\overline{W}C_j^n}{2} - \frac{\sqrt{15}}{15}\left(\overline{W}C_j^n - 2\cdot\overline{W}R_j^n\right) = d_j^n\right\},$$

then

$$\overline{W}R_j^n > \tfrac{1}{2}\overline{W}C_j^n \Rightarrow \left\{x_{MI} = x_j + \tfrac{\Delta x}{3}\left(\tfrac{2\cdot\overline{W}R_j^n - \overline{W}C_j^n}{2\cdot d_j^n - \overline{W}C_j^n}\right) < x_j - \tfrac{\Delta x}{2}\right\},$$

$$\overline{W}R_j^n < \tfrac{1}{2}\overline{W}C_j^n \Rightarrow \left\{x_{MI} = x_j + \tfrac{\Delta x}{3}\left(\tfrac{2\cdot\overline{W}R_j^n - \overline{W}C_j^n}{2\cdot d_j^n - \overline{W}C_j^n}\right) > x_j + \tfrac{\Delta x}{2}\right\}.$$

On the other hand,

$$\overline{W}R_j^n > \tfrac{1}{2}\overline{W}C_j^n \Rightarrow \left\{\tfrac{dq_j(x_j\pm\Delta x/2;\overline{w}^n)}{dx} \ge 0 \Longleftrightarrow d_j^n \ge \tfrac{1}{2}\left(4\cdot\overline{W}R_j^n - 3\cdot\overline{W}C_j^n\right)\right\},$$

$$\overline{W}R_j^n < \tfrac{1}{2}\overline{W}C_j^n \Rightarrow \left\{\tfrac{dq_j(x_j\pm\Delta x/2;\overline{w}^n)}{dx} \ge 0 \Longleftrightarrow d_j^n \ge \tfrac{1}{2}\left(\overline{W}C_j^n - 4\cdot\overline{W}R_j^n\right)\right\}.$$

Given that we have supposed that $|\frac{\overline{W}R_j^n}{\overline{W}C_j^n} - \frac{1}{2}| \le \frac{15-\sqrt{15}}{28}$, then d_j^n verifies the latter inequalities, and thus $q_j(x;\overline{w}^n)$ is monotonically increasing in I_j. ◻

3.3. Conditions in cells with extrema points. In order to guarantee that $q_j(x;\overline{w}^n)$ has the same shape as the cell-averages $\overline{w}_j^n$ in the domain I_j, we add these requirements to those used in the previous section:

1. $q_j(x;\overline{w}^n)$ has a maximum in I_j if and only if $\overline{w}_{j-1}^n < \overline{w}_j^n > \overline{w}_{j+1}^n$.
2. $q_j(x;\overline{w}^n)$ has a minimum in I_j if and only if $\overline{w}_{j-1}^n > \overline{w}_j^n < \overline{w}_{j+1}^n$.

On the other hand, the definition of θ_j^n that we will use later in (3.13) requires that the following properties are satisfied:

1. If $\overline{w}_{j-1}^n < \overline{w}_j^n > \overline{w}_{j+1}^n$, then $q_j\left(x_j - \frac{\Delta x}{2};\overline{w}^n\right) \ge \frac{1}{2}\left(\overline{w}_{j-1}^n + \overline{w}_j^n\right)$ and $q_j(x_j + \frac{\Delta x}{2};\,\overline{w}^n) \ge \frac{1}{2}\left(\overline{w}_j^n + \overline{w}_{j+1}^n\right)$.
2. If $\overline{w}_{j-1}^n > \overline{w}_j^n < \overline{w}_{j+1}^n$, then $q_j\left(x_j - \frac{\Delta x}{2};\overline{w}^n\right) \le \frac{1}{2}\left(\overline{w}_{j-1}^n + \overline{w}_j^n\right)$ and $q_j(x_j + \frac{\Delta x}{2};\,\overline{w}^n) \le \frac{1}{2}\left(\overline{w}_j^n + \overline{w}_{j+1}^n\right)$.

According to the notation given in (3.8) and (3.10), supposing that

$$(3.11)\qquad ds4_j^n = \frac{1}{6}\left(5\cdot\overline{W}C_j^n - 4\cdot\overline{W}R_j^n\right),\ ds5_j^n = \frac{1}{6}\left(4\cdot\overline{W}R_j^n + \overline{W}C_j^n\right),$$

we define d_j^n in cells with extrema points in the following way:

(B) If $\overline{w}_{j-1}^n < \overline{w}_j^n > \overline{w}_{j+1}^n$ (the cell averages have a maximum), then the following hold:

(B1) If $\overline{W}C2_j^n = 2\cdot\overline{W}C_j^n$, then $d_j^n = ds_j^n \equiv \frac{2}{3}\overline{W}C_j^n - \frac{1}{12}\overline{W}C2_j^n = \frac{1}{2}\overline{W}C_j^n$.

(B2) If $\overline{W}C2_j^n < 2 \cdot \overline{W}C_j^n$, then $d_j^n = \text{Min}\left\{ds2_j^n,\ ds4_j^n,\ ds_j^n\right\}$.
(B3) If $\overline{W}C2_j^n > 2 \cdot \overline{W}C_j^n$, then $d_j^n = \text{Max}\left\{ds3_j^n,\ ds5_j^n,\ ds_j^n\right\}$.
(C) If $\overline{w}_{j-1}^n > \overline{w}_j^n < \overline{w}_{j+1}^n$ (the cell averages have a minimum), then the following hold:
(C1) If $\overline{W}C2_j^n = 2 \cdot \overline{W}C_j^n$, then $d_j^n = ds_j^n = \frac{2}{3}\overline{W}C_j^n - \frac{1}{12}\overline{W}C2_j^n = \frac{1}{2}\overline{W}C_j^n$.
(C2) If $\overline{W}C2_j^n < 2 \cdot \overline{W}C_j^n$, then $d_j^n = \text{Min}\left\{ds3_j^n,\ ds5_j^n,\ ds_j^n\right\}$.
(C3) If $\overline{W}C2_j^n > 2 \cdot \overline{W}C_j^n$, then $d_j^n = \text{Max}\left\{ds2_j^n,\ ds4_j^n,\ ds_j^n\right\}$.

Observation 4. Similar reasoning to that used in the proof of Theorem 3.1 allows us to prove that with this definition of d_j^n, if $\overline{w}_{j-1}^n < \overline{w}_j^n > \overline{w}_{j+1}^n$, then $q_j(x;\overline{w}^n)$ has a maximum at $\left[x_j - \frac{\Delta x}{2}, x_j + \frac{\Delta x}{2}\right]$, verifying these two relations:

$$q_j\left(x_j - \frac{\Delta x}{2};\overline{w}^n\right) \geq \frac{1}{2}\left(\overline{w}_{j-1}^n + \overline{w}_j^n\right), \quad q_j\left(x_j + \frac{\Delta x}{2};\overline{w}^n\right) \geq \frac{1}{2}\left(\overline{w}_j^n + \overline{w}_{j+1}^n\right).$$

Similarly, if we suppose that $\overline{w}_{j-1}^n > \overline{w}_j^n < \overline{w}_{j+1}^n$, then we can verify that $q_j(x;\overline{w}^n)$ has a minimum at $\left[x_j - \frac{\Delta x}{2}, x_j + \frac{\Delta x}{2}\right]$, verifying the following conditions:

$$q_j\left(x_j - \frac{\Delta x}{2};\overline{w}^n\right) \leq \frac{1}{2}\left(\overline{w}_{j-1}^n + \overline{w}_j^n\right), \quad q_j\left(x_j + \frac{\Delta x}{2};\overline{w}^n\right) \leq \frac{1}{2}\left(\overline{w}_j^n + \overline{w}_{j+1}^n\right).$$

3.4. Removing the spurious extrema of $w(x,t^n)$ at points $x_j + \Delta x/2$. To obtain a nonoscillatory reconstruction we will add some additional requirements for the calculation of $R_j(x,\overline{w}^n)$:

$$\begin{aligned}&\text{(a)}\, \overline{w}(x_j,t^n) > \overline{w}(x_{j+1},t^n) \Rightarrow \left(R_j(x_j+\Delta x/2;\overline{w}^n) \geq R_{j+1}(x_j+\Delta x/2;\overline{w}^n)\right),\\ &\text{(b)}\, \overline{w}(x_j,t^n) < \overline{w}(x_{j+1},t^n) \Rightarrow \left(R_j(x_j+\Delta x/2;\overline{w}^n) \leq R_{j+1}(x_j+\Delta x/2;\overline{w}^n)\right),\\ &\text{(c)}\, \overline{w}(x_j,t^n) = \overline{w}(x_{j+1},t^n) \Rightarrow \left(R_j(x_j+\Delta x/2;\overline{w}^n) = R_{j+1}(x_j+\Delta x/2;\overline{w}^n)\right).\end{aligned} \tag{3.12}$$

These properties together to those viewed in sections 3.2 and 3.3 have been defined so that the resulting reconstruction polynomial $w(x,t^n)$, defined in (2.3), presents a nonoscillatory nature in the sense that the number of extrema of $w(x,t^n)$ does not exceed the number shown in the function $\sum_{j=1}^{NX}\overline{w}_j^n\chi_j(x)$. The nonincreasing number of extrema implies convergence along the lines of Liu and Tadmor [20].

To verify (3.12), Liu and Osher [19] consider the modification of the form

$$R_j(x;\overline{w}^n) \equiv \theta_j^n q_j(x;\overline{w}^n) + (1-\theta_j^n)\overline{w}_j^n, \tag{3.13}$$

where $\theta_j^n \in [0,1]$. The algorithm that allows us to obtain θ_j^n is described in detail in Liu and Osher [19], although in that reference it is only used when $q_j(x;\overline{w}^n)$ is a conservative parabola. Notice that the value of θ_j^n that appears in formula (3.13) takes a value equal to 1 in all the cells with extrema points (see Liu and Osher [19]). Conditions given in section 3.3 avoid the development of spurious extrema of $R_j(x;\overline{w}^n)$ in the endpoints of an interval with a local maximum or a local minimum.

Remark 1. Parameter θ_j^n used in (3.13) is defined in Liu and Osher [19] so that $(1-\theta_j^n)$ is proportional to the interface jump $q_{j+1}\left(x_j + \frac{\Delta x}{2};\overline{w}^n\right) - q_j\left(x_j + \frac{\Delta x}{2};\overline{w}^n\right)$. If $d_j^n = ds_j^n$ (given by (3.6)), then the reconstruction is fourth-order accurate. As a consequence of the fourth-order accuracy in polynomials $q_j(x;\overline{w}^n)$ and $q_{j+1}(x;\overline{w}^n)$, the size of the interface jump, and consequently of $(1-\theta_j^n)$, is of order $O((\Delta x)^4)$. In this way, the definition of $R_j(x;\overline{w}^n)$ given in (3.13) still verifies the properties in (2.4).

However, the definition of d_j^n introduced in cases (A3.1), (A3.2.1), (A3.2.2), (B2), (B3), (C2), and (C3) may cause the value of $(1-\theta_j^n)$ to not be of order $O((\Delta x)^4)$ in a small number of cells, especially near a local maximum, a local minimum, or a discontinuity. For example, if $\overline{W}R_j^n \approx \overline{W}C_j^n$ or $\overline{W}R_j^n \approx 0$ and $S_j^n > 0$, then d_j^n can be closer to $\overline{W}C_j^n/2$ (which is the slope of the parabola) than to ds_j^n. Moreover, in case (A3.2.2) d_j^n can be equal to $\overline{W}C_j^n/2$. In order to achieve the experimental fourth-order of accuracy in some experiments with smooth solutions we will give a special treatment of the cells that are near extrema (see conditions (4.2)). In fact, one should require that $d_j^n/ds_j^n = 1 + O((\Delta x)^3)$ on smooth solutions.

Remark 2. $R_j(x;\overline{w}^n)$ defined in (3.13), with $q_j(x:\overline{w}^n)$ described in subsections 3.1, 3.2, and 3.3, has the same shape as the cell-averages $\{\overline{w}_{j-1}^n, \overline{w}_j^n, \overline{w}_{j+1}^n\}$.

3.5. Definition of slopes d_j^n in (3.2) so that $p_j(x;f^n)$ fulfills conditions given in sections 3.2 and 3.3. A few modifications are needed to compute the nonoscillatory reconstruction from pointvalues for the flux $f_j^n = f(w_j^n)$ which is needed in the Runge–Kutta method with natural continuous extension described in Levy, Puppo, and Russo [16]. According to (3.2) the degree-three polynomial from the pointvalues $f_k^n,\ k \in \{j-2, j-1, j, j+1, j+2\}$, is given by

$$p_j(x;f^n) = f_j^n + d_j^n \cdot \left(\frac{x-x_j}{\Delta x}\right) + \left(\frac{f_{j-1}^n - 2f_j^n + f_{j+1}^n}{2}\right) \cdot \left(\frac{x-x_j}{\Delta x}\right)^2 + \left(\frac{-f_{j-1}^n + f_{j+1}^n - 2d_j^n}{2}\right) \cdot \left(\frac{x-x_j}{\Delta x}\right)^3. \tag{3.14}$$

To ensure that $p_j(x;f^n)$ fulfills the requirements of sections 3.2 and 3.3, we define d_j^n in the same way as in those sections with the exceptions that

$$ds1_j^n = 0,\ ds2_j^n = \frac{1}{2}\left(WC_j^n - 8 \cdot WR_j^n\right),\ ds3_j^n = \frac{1}{2}\left(8 \cdot WR_j^n - 7 \cdot WC_j^n\right), \tag{3.15}$$

$$C1 = \frac{\sqrt{3}}{6},\ C2 = \frac{6}{12+\sqrt{3}},\ WC_j^n = f_{j+1}^n - f_{j-1}^n,\ WR_j^n = f_{j+1}^n - f_j^n, \tag{3.16}$$

and cell-averages $\overline{w}_j^n$ are substituted by pointvalues f_j^n. The evaluation of $\partial f/\partial x$ in the Runge–Kutta step is performed by $\theta_j^n \frac{dp_j(x;f^n)}{dx}$, where θ_j^n is defined as in Liu and Osher [19]. Thus, we maintain high accuracy and control over oscillations.

4. Numerical experiments. In order to verify the behavior and accuracy of the numerical schemes that are presented in this paper, several test-type problems with known analytical solution are solved next. Time integrals are performed by a Taylor expansion (Taylor-upwind and Taylor-central schemes) or by the fourth-order Runge–Kutta method with natural continuous extensions developed in Levy, Puppo, and Russo [16] (RK-NCE-central scheme). In this last case the reconstruction defined in section 3.5 will also be used.

Problem 1. We solve the linear transport equation

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial u(x,t)}{\partial x} = 0, \quad -1 \le x \le 1, \tag{4.1}$$

subject to 2-periodic initial data, $u(x,0) = u_0(x)$. To verify the accuracy of the numerical schemes, different $u_0(x)$ functions have been used.

TABLE 4.1
Linear transport equation (4.1) *with* $u_0(x) = \sin(\pi x)$. *Errors at* $T = 10$.

(a) Taylor-upwind scheme, with $\Delta t = 0.8\Delta x$.

NX	L^1 error	L^1 order	L^∞ error	L^∞ order
40	3.071427 10^{-5}	4.14	2.498590 10^{-5}	4.16
80	1.746260 10^{-6}	4.05	1.396909 10^{-6}	4.06
160	1.056129 10^{-7}	4.02	8.374969 10^{-8}	4.02
320	6.529029 10^{-9}		5.152456 10^{-9}	

(b) Taylor-central scheme, with $\Delta t = 0.4\Delta x$.

NX	L^1 error	L^1 order	L^∞ error	L^∞ order
40	5.558861 10^{-5}	4.10	7.473019 10^{-5}	4.02
80	3.231451 10^{-6}	4.14	4.594014 10^{-6}	3.86
160	1.836719 10^{-7}	4.09	3.153211 10^{-7}	3.87
320	1.076991 10^{-8}		2.155580 10^{-8}	

(c) RK-NCE-central scheme, with $\Delta t = 0.25\Delta x$.

NX	L^1 error	L^1 order	L^∞ error	L^∞ order
40	1.531422 10^{-4}	4.18	1.689413 10^{-4}	4.35
80	8.423959 10^{-6}	4.06	8.305782 10^{-6}	4.02
160	5.053152 10^{-7}	4.06	5.130312 10^{-7}	3.97
320	3.034160 10^{-8}		3.280898 10^{-8}	

The first function is $u_0(x) = \sin(\pi x)$. Table 4.1 shows the errors and the experimental order of accuracy in L^1 and L^∞ norms at time $T = 10$. NX indicates the total number of cells so that the step size $\Delta x = 2/NX$. Using a Taylor expansion for the time evolution, we have selected a time step so that $\Delta t = 0.8\Delta x$ in the upwind scheme, whereas in the central scheme $\Delta t = 0.4\Delta x$. When we use a RK-NCE-central scheme, $\Delta t = 0.25\Delta x$ as in Levy, Puppo, and Russo [16]. Table 4.1 shows that numerical schemes described in this paper are about fourth-order accuracy in L^1 and L^∞ norms, which is an improvement over the schemes described in Liu and Osher [19] and Liu and Tadmor [20], which are third-order schemes.

The second initial condition chosen is $u_0(x) = \sin^4(\pi x)$. Table 4.2 shows the errors in L^1 and L^∞ norms at time $T = 10$. The schemes presented here maintain the fourth-order accuracy, even with finer grids, without the need of satisfying the local maximum principle described in Liu and Osher [19]. The nonconsideration of that local maximum principle implies that the θ_j that appear in formula (3.13) take a value equal to 1 in all the cells with extrema points (see Liu and Osher [19]). To improve the accuracy of the numerical schemes presented in this paper, in the results shown in Table 4.2 we have added two additional requirements (see Remark 1):

$$\begin{aligned} &\text{If } \overline{w}^n_{j-1} < \overline{w}^n_j > \overline{w}^n_{j+1} \Rightarrow d^n_{j-1} = ds^n_{j-1},\ d^n_{j+1} = ds^n_{j+1}. \\ &\text{If } \overline{w}^n_{j-1} > \overline{w}^n_j < \overline{w}^n_{j+1} \Rightarrow d^n_{j-1} = ds^n_{j-1},\ d^n_{j+1} = ds^n_{j+1}. \end{aligned} \tag{4.2}$$

Thus, we avoid the slope d^n_j taking a value close to $\overline{W}C^n_j/2$ in the neighboring cells to those containing the extrema points of the solution. Remember that with such a slope, $q_j(x; \overline{w}^n)$ coincides with a conservative quadratic polynomial. As mentioned in Remark 1, parameter θ^n_j is defined in such a way that $(1 - \theta^n_j)$ is proportional to the interface jump of the cell centered in x_j. Conditions (4.2) cause the interface jump to be lower at the boundary of cells with extrema points.

The third initial condition is a discontinuous 2-periodic function that was used in Balsara and Shu [2]. This is a severe problem since it consists of a combination of functions that are not smooth, with other ones, which are smooth, but with a high

Table 4.2
Linear transport equation (4.1) *with* $u_0(x) = \sin^4(\pi x)$. *Errors at* $T = 10$.

(a) Taylor-upwind scheme, with $\Delta t = 0.8\Delta x$.

NX	L^1 error	L^1 order	L^∞ error	L^∞ order
80	2.430672 10^{-4}	4.16	2.937589 10^{-4}	4.17
160	1.362647 10^{-5}	4.04	1.636374 10^{-5}	3.93
320	8.262462 10^{-7}	4.01	1.075533 10^{-6}	4.06
640	5.110279 10^{-8}		6.461631 10^{-8}	

(b) Taylor-central scheme, with $\Delta t = 0.4\Delta x$.

NX	L^1 error	L^1 order	L^∞ error	L^∞ order
80	3.499147 10^{-4}	4.14	5.062172 10^{-4}	4.49
160	1.977973 10^{-5}	4.06	2.258267 10^{-5}	3.96
320	1.189089 10^{-6}	3.99	1.454815 10^{-6}	4.06
640	7.460379 10^{-8}		8.713768 10^{-8}	

(c) RK-NCE-central scheme, with $\Delta t = 0.25\Delta x$.

NX	L^1 error	L^1 order	L^∞ error	L^∞ order
80	1.052742 10^{-3}	4.15	1.637417 10^{-3}	4.60
160	5.930366 10^{-5}	4.07	6.747230 10^{-5}	4.02
320	3.535521 10^{-6}	3.98	4.154436 10^{-6}	4.03
640	2.237979 10^{-7}		2.539025 10^{-7}	

gradient in zones close to the peaks. The initial condition is given by

$$(4.3)\quad u_0(x) = \begin{cases} \frac{1}{6}\left(G(x,\beta,z-\delta) + G(x,\beta,z+\delta) + 4G(x,\beta,z)\right), & -0.8 \le x \le -0.6, \\ 1, & -0.4 \le x \le -0.2, \\ 1 - |10\,(x-0.1)|, & 0 \le x \le 0.2, \\ \frac{1}{6}\left(F(x,\alpha,a-\delta) + F(x,\alpha,a+\delta) + 4F(x,\alpha,a)\right), & 0.4 \le x \le 0.6, \\ 0 & \text{otherwise} \end{cases}$$

defined as

$$(4.4)\qquad G(x,\beta,z) = e^{-\beta(x-z)^2}, \quad F(x,\alpha,a) = \sqrt{\operatorname{Max}\left(1-\alpha^2(x-a)^2, 0\right)}.$$

The constants that appear in (4.3) and (4.4) are given by

$$(4.5)\qquad a = 0.5;\ z = -0.7;\ \delta = 0.005;\ \alpha = 10;\quad \beta = \frac{\log(2)}{36\delta^2}.$$

Figure 4.1 shows the numerical results obtained at time $T = 20$, with the Taylor-central scheme developed in this paper, comparing the numerical solution with the analytical solution which is represented by a continuous line. Unlike the solutions presented in Balsara and Shu [2], here we have considered a coarser grid with $NX = 500$. This shows the greater accuracy of our scheme, in comparison with the conservative quadratic polynomial developed in Liu and Osher [19], in particular at the peaks of the Gaussian curve and in the triangle. In addition, the profiles are more symmetrical than those computed in Levy, Puppo, and Russo [16], and the values of the numerical solution are bounded by the maximum and minimum of the initial condition despite not using the maximum principle property given in Liu and Osher [19]. This is a condition that is not fulfilled when $q_j(x;\overline{w}^n)$ is replaced by the two-degree polynomial used in this reference. Previous remarks for the Taylor-central scheme are also valid for the Taylor-upwind and RK-NCE-central schemes.

The last initial condition is given by

$$(4.6)\qquad u_0(x+0.5) = \begin{cases} -x\sin\left(\frac{3}{2}\pi x^2\right) & \text{if } -1 < x < -\frac{1}{3}, \\ |\sin(2\pi x)| & \text{if } |x| \le \frac{1}{3}, \\ 2x - 1 - \sin(3\pi x)/6, & \text{if } \frac{1}{3} < x \le 1, \end{cases}$$

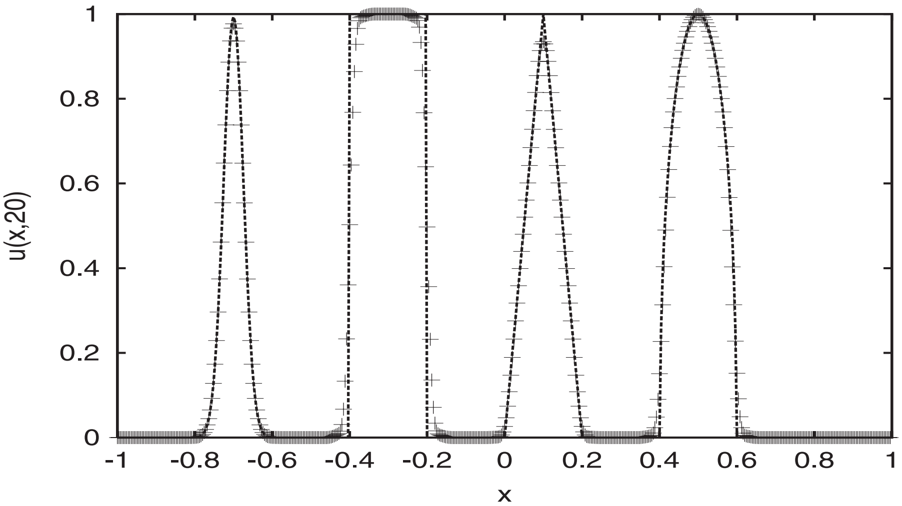


Fig. 4.1. *Numerical and analytical solutions of problem 1 at $T = 20$ with $u_0(x)$ defined by (4.3)–(4.5), considering a grid with $NX = 500$. We have used the Taylor-central scheme, with $\Delta t = 0.45\Delta x$.*

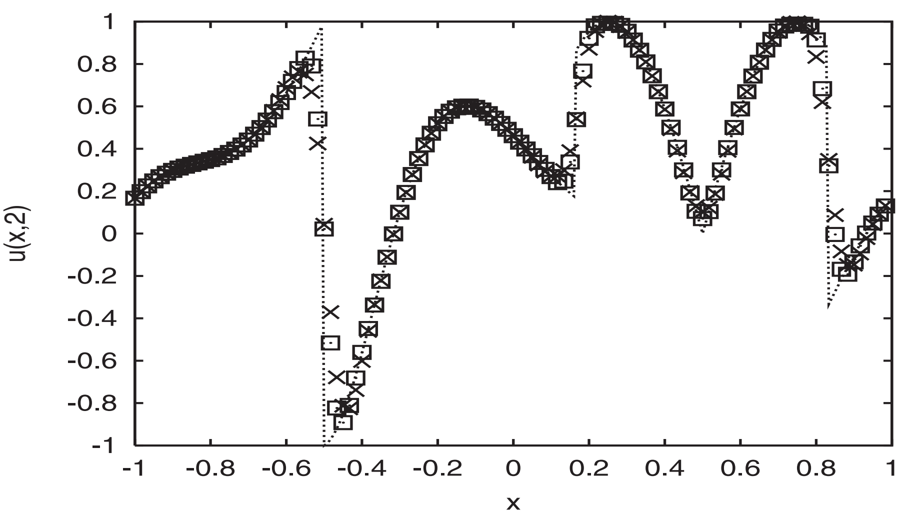


Fig. 4.2. *Numerical and analytical solutions of Problem 1 at $T = 2$ with $u_0(x)$ defined by (4.6), considering $NX = 120$ and $\Delta t = 0.4\Delta x$. The solution marked with $\boxdot$ has been computed by our Taylor-central scheme. The solution marked with $\times$ is the solution obtained considering that $q_j(x;\overline{w}^n)$ coincides with the quadratic polynomial of Liu and Osher* [19] *and Liu and Tadmor* [20].

supposing that it extends to the entire 2-period domain.

Condition (4.6) consists of a function highly discontinuous, used in the numerical experiments developed in Harten [9]. Figure 4.2 shows the results obtained at $T = 2$, with $NX = 120$, comparing the results obtained with our Taylor-central scheme and those in which $q_j(x;\overline{w}^n)$ is the conservative two-degree polynomial used in Liu and Osher [19] and Liu and Tadmor [20]. It can be observed that the greater accuracy of our schemes is especially noted around the discontinuities. On the other hand, the

TABLE 4.3
Errors in the resolution of Burgers's equation with conditions (4.7) *at* $T = 0.3$.

(a) Taylor-upwind scheme, with $\Delta t = 0.6\Delta x$.

NX	L^1 error	L^1 order	L^∞ error	L^∞ order
80	$2.551210\ 10^{-6}$	4.02	$1.018463\ 10^{-5}$	4.06
160	$1.567119\ 10^{-7}$	4.01	$6.108933\ 10^{-7}$	3.94
320	$9.734252\ 10^{-9}$	4.00	$3.988465\ 10^{-8}$	4.07
640	$6.090892\ 10^{-10}$	3.95	$2.370593\ 10^{-9}$	3.37
1280	$3.947035\ 10^{-11}$		$2.291545\ 10^{-10}$	

(b) Taylor-central scheme, with $\Delta t = 0.33\Delta x$.

NX	L^1 error	L^1 order	L^∞ error	L^∞ order
80	$1.536227\ 10^{-6}$	4.18	$8.083824\ 10^{-6}$	4.22
160	$8.454566\ 10^{-8}$	4.10	$4.340597\ 10^{-7}$	4.18
320	$4.916180\ 10^{-9}$	4.07	$2.392437\ 10^{-8}$	4.10
640	$2.935978\ 10^{-10}$	3.99	$1.393063\ 10^{-9}$	4.13
1280	$1.846273\ 10^{-11}$		$7.967693\ 10^{-11}$	

(c) RK-NCE-central scheme, with $\Delta t = 0.18\Delta x$.

NX	L^1 error	L^1 order	L^∞ error	L^∞ order
80	$2.703482\ 10^{-6}$	4.18	$1.875872\ 10^{-5}$	4.22
160	$1.496377\ 10^{-7}$	4.21	$1.006314\ 10^{-6}$	4.24
320	$8.089214\ 10^{-9}$	4.17	$5.322066\ 10^{-8}$	4.33
640	$4.495486\ 10^{-10}$	4.12	$2.641527\ 10^{-9}$	4.13
1280	$2.589858\ 10^{-11}$		$1.506755\ 10^{-10}$	

numerical solution is delimited by the maximum and minimum of the initial condition without the need of satisfying the local maximum principle of Liu and Osher [19].

Problem 2. Burgers' equation is solved with 2-periodic initial data:

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial\left(\frac{1}{2}u^2(x,t)\right)}{\partial x} = 0, \quad -1 \le x \le 1, \quad u\,(x,0) = 1 + \frac{1}{2}\sin(\pi x). \tag{4.7}$$

Recall that the analytical solution of this problem is smooth up to the critical time $T = 2/\pi$. Liu and Osher [19] and Liu and Tadmor [20] show the results obtained using a parabolic reconstruction at $T = 0.3$. Table 4.3 presents the numerical errors obtained with our schemes, together with the experimental order of accuracy at $T = 0.3$. Our schemes (upwind and central) have an order of accuracy which is about 4 in both L^1 and L^∞ norms. However, the maximum order obtained with the schemes described by Liu and Osher [19] (Taylor-upwind scheme, $\Delta t = 0.6\Delta x$) and Liu and Tadmor [20] (Taylor-central scheme, $\Delta t = 0.33\Delta x$) is lower than 2.3 in the L^∞ norm and lower than 2.87 in the L^1 norm. In the RK-NCE-central scheme we have chosen $\Delta t = 0.18\Delta x$ as in Levy, Puppo, and Russo [16].

At $T = 1.1$ the analytical solution of problem (4.7) develops a discontinuity. Our numerical scheme maintains an order of accuracy of about 4 when the errors are calculated at a distance equal to 0.1 away from the discontinuity. Figure 4.3(a) shows the result obtained with our Taylor-central scheme. Like in the scheme developed in Liu and Tadmor [20], the numerical solution is not bounded by the maximum and minimum of the analytical solution. In order to obtain this property it is necessary to add the maximum principle requirement described in Liu and Osher [19], as we can see in Figure 4.3(b). However, without the condition of maximum principle, the numerical solution retains results of the same quality as the analytical solution. Previous remarks are also valid for the Taylor-upwind and RK-NCE-central schemes.

Problem 3. Here we apply the schemes developed in this paper to Buckley–

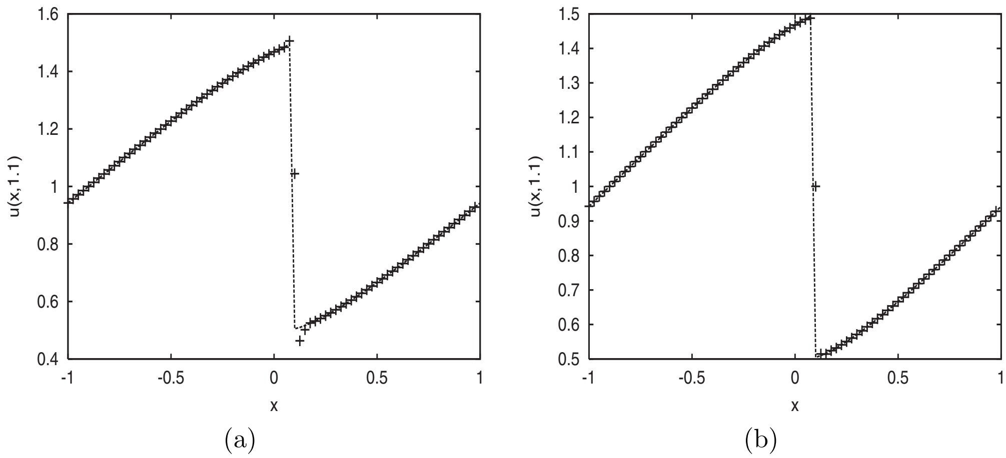


FIG. 4.3. *Numerical and analytical solutions of Problem* 2 *at* $T = 1.1$*, with our Taylor-central scheme, considering a grid with* $NX = 80$ *and* $\Delta t = 0.33\Delta x$*.* (a) *Not using the local maximum principle.* (b) *Using the local maximum principle.*

Leverett's problem, whose flux is nonconvex:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad -1 \leq x \leq 1, \quad f(u) = \frac{4u^2}{4u^2 + (1-u)^2} \tag{4.8}$$

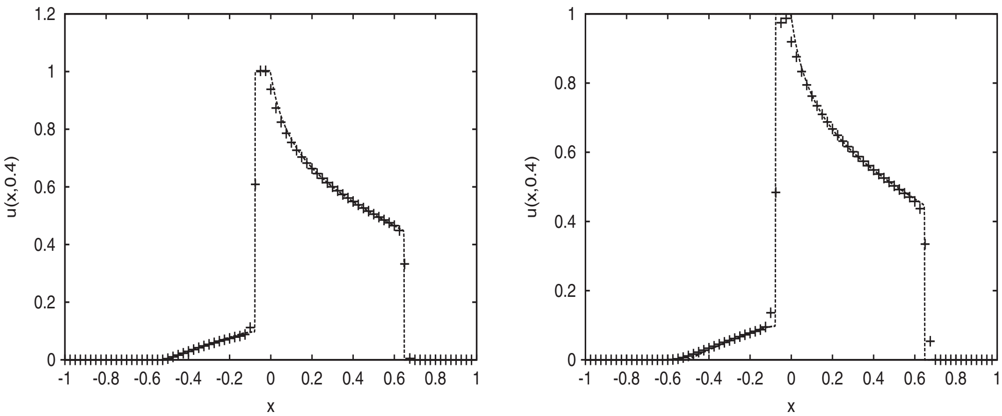
subject to the initial condition

$$u_0(x) = \begin{cases} 1 & x \in [-0.5, 0], \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

Similarly to Liu and Osher [19] and Jiang et al. [11], we have computed the solution at $T = 0.4$ with our Taylor-upwind and central schemes. Figure 4.4 shows the results obtained with $NX = 80$. In contrast to the scheme described in Liu and Osher [19], our Taylor-upwind scheme presents instabilities in the solution of the problem (4.8)–(4.9) for $\Delta t = 0.3\Delta x$. However, it presents very accurate solutions when $\Delta t = 0.25\Delta x$. Moreover, the condition of the local maximum principle described in Liu and Osher [19] has not been necessary, as shown in Figure 4.4(a). The central schemes described in this paper provide smoother solutions than the upwind scheme for the resolution of the problem under study (Figures 4.4(a)–4.4(b)), although the three schemes present a similar behavior.

Euler equations of gas dynamics. We test our schemes on the system of Euler equations of gas dynamics for a gas with $\gamma = 1.4$. We consider a problem with smooth analytical solution and the two Riemann problems studied in Liu and Tadmor [20]. The variables ρ, m, E denote the density, momentum, and total energy per unit volume, respectively. Moreover, p denotes the pressure and v denotes the velocity.

Problem 4. The initial condition is set to be $\rho(x, 0) = 1 + 0.2\sin(\pi x)$, $v(x, 0) = 1$, $p(x, 0) = 1$, with 2-periodic boundary conditions, $-1 \leq x \leq 1$. The exact solution is $\rho(x, t) = 1 + 0.2\sin(\pi(x - t))$, $v = 1$, $p = 1$. We compute the solution at $T = 2$ as in Qiu and Shu [22], using our RK-NCE-central scheme with the componentwise reconstruction described in this paper. Table 4.4 shows the results obtained considering $\Delta t = 0.1\Delta x$. We can see that our scheme achieves its designed order of accuracy.



(a) (b)

FIG. 4.4. *Numerical and analytical solutions of Problem* 3 *at* $T = 0.4$*, using a grid with* $NX = 80$. (a) *Taylor-upwind scheme with* $\Delta t = 0.25\Delta x$. (b) *Taylor-central scheme with* $\Delta t = 0.1\Delta x$.

TABLE 4.4
Errors of density in the resolution of Problem 4 *at* $T = 2$.

RK-NCE-central scheme, with $\Delta t = 0.1\Delta x$.

NX	L^1 error	L^1 order	L^∞ error	L^∞ order
40	9.910941 10^{-6}	4.53	7.147074 10^{-6}	4.53
80	3.870207 10^{-7}	4.26	3.083984 10^{-7}	4.27
160	2.020669 10^{-8}	4.10	1.601040 10^{-8}	4.11
320	1.178203 10^{-9}	4.04	9.296230 10^{-10}	4.04
640	7.181752 10^{-11}		5.656586 10^{-11}	

Problem 5. Shock tube problem with Sod's initial data [24]:

$$\begin{cases} (\rho_l, m_l, E_l) = (1, 0, 2.5), & x < 0.5, \\ (\rho_r, m_r, E_r) = (0.125, 0, 0.25), & x > 0.5. \end{cases}$$

Problem 6. Shock tube problem with the Lax's initial data [14]:

$$\begin{cases} (\rho_l, m_l, E_l) = (0.445, 0.311, 8.928), & x < 0.5, \\ (\rho_r, m_r, E_r) = (0.5, 0, 1.4275), & x > 0.5. \end{cases}$$

In Problems 5 and 6 the computational domain is $[0, 1]$. We integrate the equations to $T = 0.16$, i.e., before the perturbations reach the boundary of the computational region (free flow boundary conditions). We compute the numerical solution with our RK-NCE-central scheme, using the componentwise reconstruction described in this paper. In Figure 4.5 we plot the computed solution with $NX = 200$ grid points as in Liu and Tadmor [20]. We observe the improved resolution in comparison to the corresponding third-order central results of that reference. However, our solutions present more oscillations, which is in agreement with what is commented on in [22]. Qiu and Shu [22] conclude that the componentwise central WENO scheme will become more oscillatory when the order of accuracy increases. Qiu and Shu [22] also observe that the oscillations disappear when the reconstruction is performed on characteristic variables. It is conceivable to expect that the same thing will happen with the new reconstruction proposed in this paper. This will be explored in some future work.

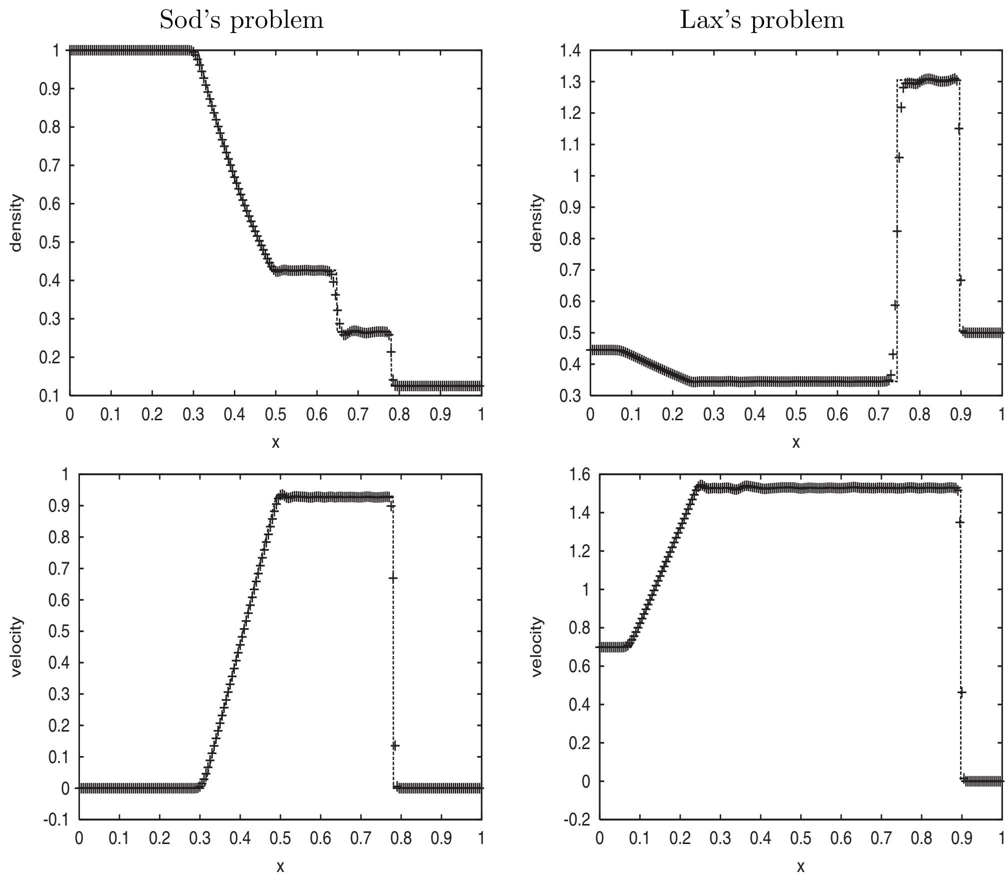


FIG. 4.5. *Numerical and analytical solutions of Euler equations for Problems* 5 *and* 6 *at time* $T = 0.16$*, with our RK-NCE-central scheme, considering a grid with* $NX = 200$*. We have considered that* $\Delta t = 0.1\Delta x$ *in the Sod problem and* $\Delta t = 0.09\Delta x$ *in the Lax problem.*

5. Conclusions. This paper presents a new fourth-order nonoscillatory reconstruction procedure for upwind and central schemes that solves hyperbolic conservation laws in one spatial dimension, improving the accuracy of the schemes developed in Liu and Osher [19] and Liu and Tadmor [20]. We have proved that our schemes are *number of extrema decreasing* and this implies convergence along the lines of Liu and Tadmor [20]. Numerical experiments have shown that our schemes are fourth-order accurate, conservative, and nonoscillatory, presenting good behavior without the need of satisfying the local maximum principle described in Liu and Osher [19]. Future research will extend these schemes to several spatial variables. We also may study the linear stability of these schemes by a procedure similar to that developed in Bianco, Puppo, and Russo [3].

Acknowledgments. We express our gratitude to the four anonymous reviewers for their helpful comments. We thank Dr. Arturo Hidalgo López for his suggestions. We would like to thank the Foreign Language Coordination Office at the Polytechnic University of Valencia for their help in translating this paper.

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